GENERALIZED CLASS OF $p-$VALENT FUNCTIONS WITH HIGHER ORDER DERIVATIVES AND NEGATIVE COEFFICIENTS

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Abstract. In this paper we introduced and studied new class $F_\theta(p,q,\gamma,\beta)$ of p-valent functions with negative coefficients. We obtained coefficients inequalities, distortion theorems, extreme points and radii of close to convexity, starlikeness and convexity for functions in this class. Also functions in this modified Hadamard products of several functions belonging to the class $F_\theta(p,q,\gamma,\beta)$ are obtained. Finally, several distortion inequalities involving fractional calculus are investigated.

Keywords: Analytic, $p-$valent functions, Hadamard product, fractional calculus operators.

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1. Introduction

Let $T_p(\theta)$ denotes the class of functions of the form:

$$f(z)=z^p-\sum_{k=1}^{\infty}a_{p+k}z^{p+k}\left(e^{i\theta}a_{p+k}\geq 0;\ |\theta|<\frac{\pi}{2};\ p\in\mathbb{N}=\{1,2,...\}\right),\tag{1}$$

which are analytic and p-valent in the unit disc $U=\{z\ :\ z\in\mathbb{C}$ and $|z|<1\}$. For $f(z)\in T_p(\theta)$, we have

$$f^{(m)}(z)=\delta(p,m)z^{p-m}-\sum_{k=1}^{\infty}\delta(k+p,m)a_{p+k}z^{p+k-m}\ (m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}),\tag{2}$$

where

$$\delta(p,m)=\frac{m!}{(p-m)!}=\begin{cases}p(p-1)...(p-q+1) & (m\neq 0),\\ 1 & (m=0).\end{cases}$$

Let $F_\theta(p,q,\gamma,\beta)$ denote the class of functions $f(z)\in T_p(\theta)$ which satisfy

$$\mathrm{Re}\left\{e^{i\theta}\left((1-\gamma)\frac{f^{(q)}(z)}{\delta(p,q)z^{p-q}}+\gamma\frac{f^{(q+1)}(z)}{\delta(p,q+1)z^{p-q-1}}\right)\right\}>\frac{\beta}{p-q},\tag{3}$$

where $0\le\frac{\beta}{p-q}<\cos\theta,\ |\theta|<\frac{\pi}{2},\ \gamma\ge 0,\ p\in\mathbb{N}, q\in\mathbb{N}_0, p>q$ *and* $z\in U$.

We note that for suitable choices of γ,θ,p and q, we obtain the following subclasses:

(1) $F_{\theta}(p,0,\gamma,\beta)=F_{p,\theta}(\gamma,\beta)\left(0\le\beta<p,\ |\theta|<\frac{\pi}{2},\ \gamma\ge 0,\ |\theta|<\frac{\pi}{2}, p\in \mathrm{N}\right)$ (see EL-Ashwah et al. [4]);

(2) $F_0(p,0,\gamma,\beta)=F_p(\gamma,\beta)\left(0\le\beta<1,\ |\theta|<\frac{\pi}{2},\ \gamma\ge 0, p\in\mathrm{N}\right)$ (see Lee et al. [5] and Aouf and Darwish [2]);

(3) $F_0(1,0,\gamma,\beta)=F(\gamma,\beta)(0\le\beta<1,\ \gamma\ge 0, p\in\mathrm{N})$ (see Bhoosnurmath and Swamy [3]);

(4) $F_{\theta}(1,0,1,\beta)=A(\theta,\beta)\left(0\le\beta<\cos\theta,\ |\theta|<\frac{\pi}{2}\right)$ (see Sekine [8]).

Also, we note that:

(1) $F_{\theta}(p,q,0,\beta)=F_{\theta}(p,q,\beta)=\left\{f\in T_p(\theta):\ \mathrm{Re}\left[e^{i\theta}\frac{f^{(q)}(z)}{\delta(p,q)z^{p-q}}\right]>\frac{\beta}{p-q}\right.$

$\left.\left(0\le\frac{\beta}{p-q}<\cos\theta,\ |\theta|<\frac{\pi}{2},\ p\in\mathrm{N},\ q\in\mathrm{N}_0,\ p>q,\ z\in U\right)\right\}$;

(2) $F_{\theta}(p,q,1,\beta)=G_{\theta}(p,q,\beta)=\left\{f\in T_p(\theta):\ \mathrm{Re}\left[e^{i\theta}\frac{f^{(q+1)}(z)}{\delta(p,q+1)z^{p-q-1}}\right]>\frac{\beta}{p-q}\right.$

$\left.\left(0\le\frac{\beta}{p-q}<\cos\theta,\ |\theta|<\frac{\pi}{2},\ p\in\mathrm{N},\ q\in\mathrm{N}_0,\ p>q,\ z\in U\right)\right\}$.

2. Cofficient estimates

Unless otherwise mentioned, we assume throughout this paper that

$$e^{i\theta}a_{p+k}\ge 0,\ 0\le\frac{\beta}{p-q}<\cos\theta,\ |\theta|<\frac{\pi}{2},\ \gamma\ge 0,\ q\in\mathrm{N}_0,\ p\in\mathrm{N} \text{ and } p>q.$$

Theorem 1. Let the function $f(z)$ be given by (1) . Then $f(z)\in F_{\theta}\left(p,q,\gamma,\beta\right)$
if and only if

$$\sum_{k=1}^{\infty} e^{i\theta}\left(p-q+\gamma k\right)\delta(p+k,q)a_{p+k} \le \left(\cos\theta-\frac{\beta}{p-q}\right)\delta(p,q+1). \qquad (4)$$

Proof. Suppose that (4) holds. It is sufficient to show that the value for

$$e^{i\theta}\left((1-\gamma)\frac{f^{(q)}(z)}{\delta\left(p,q\right)z^{p-q}}+\gamma\frac{f^{(q+1)}(z)}{\delta\left(p,q+1\right)z^{p-q-1}}\right),$$

lies in a circle centered at a point $e^{i\theta}$ whose radius is $\left(\cos\theta-\frac{\beta}{p-q}\right)\delta(p,q+1).$ Indeed, we have

$$\left|e^{i\theta}\left((1-\gamma)\frac{f^{(q)}(z)}{\delta\left(p,q\right)z^{p-q}}+\gamma\frac{f^{(q+1)}(z)}{\delta\left(p,q+1\right)z^{p-q-1}}\right)-e^{i\theta}\right|$$

$$=\left|e^{i\theta}\sum_{k=1}^{\infty}\frac{(p-q+\gamma k)\delta(p+k,q+1)}{(p+k-q)\delta(p,q+1)}a_{p+k}z^{k}\right|$$

$$\le\sum_{k=1}^{\infty}e^{i\theta}\left(p-q+\gamma k)\delta(p+k,q\right)a_{p+k}$$

$$\le\left(\cos\theta-\frac{\beta}{p-q}\right)\delta(p,q+1).$$

Conversely, assume that

$$\operatorname{Re}\left\{e^{i\theta}((1-\gamma)\frac{f^{(q)}(z)}{\delta\left(p,q\right)z^{p-q}}+\gamma\frac{f^{(q+1)}(z)}{\delta\left(p,q+1\right)z^{p-q-1}})\right\}>\frac{\beta}{p-q},$$

which is equivalent to

$$\operatorname{Re}\left\{\sum_{k=1}^{\infty}e^{i\theta}\frac{(p-q+\gamma k)\delta(p+k,q+1)}{(p+k-q)\delta(p,q+1)}a_{p+k}z^{k}\right\}<\cos\theta-\frac{\beta}{p-q}.$$

Choose values of z on the real axis so that

$$\sum_{k=1}^{\infty}e^{i\theta}\frac{(p-q+\gamma k)\delta(p+k,q+1)}{(p+k-q)\delta(p,q+1)}a_{p+k}z^{k},$$

is real. Letting $z\to 1^{-}$ along the real axis, we have

$$\sum_{k=1}^{\infty} e^{i\theta}\left(p-q+\gamma k\right)\delta(p+k,q)a_{p+k} \le \left(\cos\theta-\frac{\beta}{p-q}\right)\delta(p,q+1),$$

and hence the proof of Theorem 1 is completed.

Corollary 1. Let the function $f(z)$ defined by (1) be in the class $F_\theta\left(p,q,\gamma,\beta\right)$.
Then

$$\left|a_{p+k}\right| \le \frac{((p-q)\cos\theta-\beta)\delta(p,q)}{(p-q+k\gamma)\delta(p+k,q)}(k\ge 1). \tag{5}$$

The result is sharp for the function

$$f(z)=z^p-\frac{((p-q)\cos\theta-\beta)\delta(p,q)}{(p-q+k\gamma)\delta(p+k,q)}e^{-i\theta}z^{p+k}. \tag{6}$$

3. Distortion theorems

Theorem 2. Let the function $f(z)$ defined by (1) be in the class $F_\theta\left(p,q,\gamma,\beta\right)$.
then for $p>m,\quad z\in U,$ we have

$$\left|f^{(m)}(z)\right| \ge \left(\delta(p,m)-\frac{((p-q)\cos\theta-\beta)(p-q+1)\delta(p+1,m)}{(p+\gamma-q)(p+1)}r\right)r^{p-m}, \tag{7}$$

and

$$\left|f^{(m)}(z)\right| \le \left(\delta(p,m)+\frac{((p-q)\cos\theta-\beta)(p-q+1)\delta(p+1,m)}{(p+\gamma-q)(p+1)}r\right)r^{p-m}. \tag{8}$$

The result is sharp for the function $f(z)$ given by

$$f(z)=z^p-\frac{((p-q)\cos\theta-\beta)\delta(p,q)}{(p-q+k\gamma)\delta(p+k,q)}e^{-i\theta}z^{p+k}\ (z=\pm|z|e^{i\theta}). \tag{9}$$

Proof. It is easy to see from Theorem 1 that

$$\left(p+\gamma-q\right)\delta(p+1,q)\sum_{k=1}^{\infty}a_{p+k} \le \sum_{k=1}^{\infty}e^{i\theta}\left(p-q+\gamma k\right)\delta(p+k,q)a_{p+k}$$

$$\le\left(\cos\theta-\frac{\beta}{p-q}\right)\delta(p,q+1),$$

$$\sum_{k=1}^{\infty}a_{p+k} \le \frac{((p-q)\cos\theta-\beta)\delta(p,q)}{(p+\gamma-q)\delta(p+1,q)}. \tag{10}$$

Making use of (10) and for $|z| = r < 1,$ we have

$$\left|f^{(m)}(z)\right| \geq \delta(p,m)|z|^{p-m} - \delta(p+1,m)|z|^{p+1-m}\sum_{k=1}^{\infty} a_{p+k}$$

$$\geq \delta(p,m)|z|^{p-m} - \delta(p+1,m)\frac{((p-q)\cos\theta-\beta)\delta(p,q)}{(p+\gamma-q)\delta(p+1,q)}|z|^{p+1-m}$$

$$\geq \left(\delta(p,m) - \frac{((p-q)\cos\theta-\beta)(p-q+1)\delta(p+1,m)}{(p+\gamma-q)(p+1)} r\right) r^{p-m}, \quad (11)$$

and

$$\left|f^{(m)}(z)\right| \leq \delta(p,m)|z|^{p-m} + \delta(p+1,m)|z|^{p+1-m}\sum_{k=1}^{\infty} a_{p+k}$$

$$\leq \delta(p,m)|z|^{p-m} + \delta(p+1,m)\frac{((p-q)\cos\theta-\beta)\delta(p,q)}{(p+\gamma-q)\delta(p+1,q)}|z|^{p+1-m}$$

$$\leq \left(\delta(p,m) + \frac{((p-q)\cos\theta-\beta)(p-q+1)\delta(p+1,m)}{(p+\gamma-q)(p+1)} r\right) r^{p-m}, \quad (12)$$

which proves the assertion (7) and (8).

Putting $m = 0$ in Theorem 2, we have the following corollary.

Corollary 2. Let the function $f(z)$ defined by (1) be in the class $F_\theta\left(p,q,\gamma,\beta\right)$. Then for $|z| = r < 1,$ we have

$$|f(z)| \geq \left[1 - \frac{((p-q)\cos\theta-\beta)(p-q+1)}{(p+\gamma-q)(p+1)} r\right] r^p. \quad (13)$$

and

$$|f(z)| \leq \left[1 + \frac{((p-q)\cos\theta-\beta)(p-q+1)}{(p+\gamma-q)(p+1)} r\right] r^p. \quad (14)$$

The result is sharp.

Putting $m = 1$ in Theorem 2, we have the following corollary.

Corollary 3. Let the function $f(z)$ defined by (1) be in the class $F_\theta\left(p,q,\gamma,\beta\right)$. Then for $|z| = r < 1,$ we have

$$|f'(z)| \geq \left[p - \frac{((p-q)\cos\theta-\beta)(p-q+1)}{(p+\gamma-q)} r\right] r^{p-1}. \quad (15)$$

and

$$|f'(z)| \le \left[p - \frac{((p-q)\cos\theta - \beta)(p-q+1)}{(p+\gamma-q)} r \right] r^{p-1}. \tag{16}$$

The result is sharp.

4. Closure theorems

Let the functions $f_j(z)$ $(j=1,2,...,m)$, defined by

$$f_j(z) = z^p - \sum_{k=1}^{\infty} a_{p+k,j} z^{p+k} \left(\pi e^{i\theta} a_{p+k,j} \ge 0;\ |\theta| < \frac{\pi}{2} \right). \tag{17}$$

Theorem 3. Let the functions $f_j(z)$ $(j=1,2,...,m)$ defined by (17) be in the class $F_\theta\left(p,q,\gamma,\beta\right)$. Then the function $h(z)$ defined by

$$h(z) = z^p - \sum_{k=1}^{\infty} b_{p+k} z^{p+k}, \tag{18}$$

also belongs to the class $F_\theta\left(p,q,\gamma,\beta\right)$, where

$$b_{p+k} = \frac{1}{m} \sum_{j=1}^{m} a_{p+k,j}. \tag{19}$$

Proof. Since $f_j(z)$ $(j=1,2,...,m)$ are in the class $F_\theta\left(p,q,\gamma,\beta\right)$, it follows from Theorem 1, that

$$\sum_{k=1}^{\infty} e^{i\theta}\left(p-q+\gamma k\right)\delta(p+k,q) a_{p+k,j} \le \left(\cos\theta - \frac{\beta}{p-q} \right) \delta(p,q+1),$$

for every $j=1,2,...,m.$ Hence, we have

$$\begin{aligned}
&\sum_{k=1}^{\infty} e^{i\theta}\left(p-q+\gamma k\right)\delta(p+k,q) b_{p+k} \\
&= \sum_{k=1}^{\infty} e^{i\theta}\left(p-q+\gamma k\right)\delta(p+k,q)\left(\frac{1}{m}\sum_{j=1}^{m} a_{p+k,j} \right) \\
&= \frac{1}{m}\sum_{j=1}^{m}\left(\sum_{k=1}^{\infty} e^{i\theta}\left(p-q+\gamma k\right)\delta(p+k,q) a_{p+k,j} \right) \\
&\le \frac{1}{m}\sum_{j=1}^{m}\left(\cos\theta - \frac{\beta}{p-q} \right)\delta(p,q+1) \\
&\le \left(\cos\theta - \frac{\beta}{p-q} \right)\delta(p,q+1).
\end{aligned}$$

By Theorem 1, it follows that $h(z) \in F_\theta\left(p,q,\gamma,\beta\right)$. This completes the proof of Theorem 3.

Theorem 4. Let the functions $f_j(z)\ (j=1,2,....,m)$ defined by (17) be in the class $F_\theta\left(p,q,\gamma,\beta_j\right)$. Then the function $h(z)$ defined by

$$h(z) = z^p - \sum_{k=1}^{\infty}\left(\frac{1}{m}\sum_{j=1}^{m} a_{p+k,\,j}\right) z^{p+k}, \tag{20}$$

is in the class $F_\theta\left(p,q,\gamma,\beta\right)$, where

$$\beta = \min_{1\le j\le m}\{\beta_j\}.$$

Proof. Since $f_j(z)\ (j=1,2,....,m)$ are in the class $F_\theta\left(p,q,\gamma,\beta_j\right)$, it follows from Theorem 1, that

$$\sum_{k=1}^{\infty} e^{i\theta}\left(p-q+\gamma k\right)\delta(p+k,q)a_{p+k,\,j} \le \left(\cos\theta - \frac{\beta_j}{p}\right)\delta(p,q+1),$$

for every $j=1,2,....,m.$ Hence we have

$$\sum_{k=1}^{\infty} e^{i\theta}\left(p-q+\gamma k\right)\delta(p+k,q)\left(\frac{1}{m}\sum_{j=1}^{m} a_{p+k,\,j}\right)$$

$$= \frac{1}{m}\sum_{j=1}^{m}\left(\sum_{k=1}^{\infty} e^{i\theta}\left(p-q+\gamma k\right)\delta(p+k,q)a_{p+k,\,j}\right)$$

$$\le \frac{1}{m}\sum_{j=1}^{m}\left(\cos\theta - \frac{\beta_j}{p-q}\right)\delta(p,q+1)$$

$$\le \frac{1}{m}\sum_{j=1}^{m}\left(\cos\theta - \frac{\beta}{p-q}\right)\delta(p,q+1)$$

$$\le \left(\cos\theta - \frac{\beta}{p-q}\right)\delta(p,q+1).$$

By Theorem 1, it follows that $h(z) \in F_\theta\left(p,q,\gamma,\beta\right)$ This completes the proof of Theorem 4.

Theorem 5. Let the functions $f_j(z)\ (j=1,2,...,m)$ defined by (17) be in the class $F_\theta\left(p,q,\gamma,\beta\right)$. Then the function $h(z)$ defined by

$$h(z) = \sum_{j=1}^{m} c_j f_j(z), \tag{21}$$

is also in the class $F_\theta\left(p,q,\gamma,\beta\right),$ where

$$\sum_{j=1}^{m} c_j = 1,\ c_j \geq 0. \tag{22}$$

Proof. Assume that

$$h(z) = \sum_{j=1}^{m} c_j f_j(z)$$

$$= z^p - \sum_{k=1}^{\infty}\left(\sum_{j=1}^{m} c_j a_{p+k,j}\right) z^{p+k}. \tag{23}$$

Then it follows that

$$\sum_{k=1}^{\infty} e^{i\theta}\left(p-q+\gamma k\right)\delta(p+k,q)\left(\sum_{j=1}^{m} c_j a_{p+k,j}\right)$$

$$= \sum_{j=1}^{m} c_j \left(\sum_{k=1}^{\infty} e^{i\theta}\left(p-q+\gamma k\right)\delta(p+k,q) a_{p+k,j}\right)$$

$$\leq \left(\cos\theta - \frac{\beta}{p-q}\right)\delta(p,q+1)\sum_{j=1}^{m} c_j$$

$$\leq \left(\cos\theta - \frac{\beta}{p}\right)\delta(p,q+1).$$

By Theorem 1, it follows that $h(z) \in F_\theta\left(p,q,\gamma,\beta\right)$. This completes the proof of Theorem 5.

Theorem 6. Let $f_p(z) = z^p$ and

$$f_{p+k}(z) = z^p - \frac{((p-q)\cos\theta - \beta)\delta(p,q)}{(p-q+\gamma k)\delta(p+k,q)} e^{-i\theta} z^{p+k}\ (k \geq 1). \tag{24}$$

Then $f(z)$ is in the class $F_\theta\left(p,q,\gamma,\beta\right)$ if and only if can be expressed in the form

$$f(z) = \sum_{k=0}^{\infty} \mu_{p+k} f_{p+k}(z), \tag{25}$$

where $\mu_{p+k} \geq 0$ and $\sum_{k=0}^{\infty} \mu_{p+k} = 1.$

Proof. Assume that

$$f(z)=\sum_{k=0}^{\infty}\mu_{p+k}f_{p+k}(z)$$
$$=z^{p}-\sum_{k=1}^{\infty}\frac{((p-q)\cos\theta-\beta)\delta(p,q)}{(p-q+k\gamma)\delta(p+k,q)}e^{-i\theta}\mu_{p+k}z^{p+k}. \quad (26)$$

Then it follows that

$$\sum_{k=1}^{\infty}e^{i\theta}\left(\frac{(p-q+k\gamma)\delta(p+k,q)}{\delta(p,q+1)}\right)\left(\frac{((p-q)\cos\theta-\beta)\delta(p,q)}{(p-q+k\gamma)\delta(p+k,q)}\right)e^{-i\theta}\mu_{p+k}$$

$$=\left(\cos\theta-\frac{\beta}{p-q}\right)\sum_{k=1}^{\infty}\mu_{p+k}$$
$$=\left(\cos\theta-\frac{\beta}{p-q}\right)(1-\mu_{p})$$
$$\leq\cos\theta-\frac{\beta}{p-q},$$

which implies that $f(z)\in F_{\theta}\left(p,q,\gamma,\beta\right)$.

Conversely, assume that the function $f(z)$ defined by (1) be in the class $F_{\theta}\left(p,q,\gamma,\beta\right)$. Then

$$a_{p+k}\leq\frac{((p-q)\cos\theta-\beta)\delta(p,q)}{(p-q+k\gamma)\delta(p+k,q)}.$$

Setting

$$\mu_{p+k}=\frac{(p-q+k\gamma)\delta(p+k,q)}{((p-q)\cos\theta-\beta)\delta(p,q)}e^{i\theta}a_{p+k},$$

where

$$\mu_{p}=1-\sum_{k=1}^{\infty}\mu_{p+k},$$

we can see that $f(z)$ can be expressed in the form (25) . This completes the proof of Theorem 6.

Corollary 4. The extreme points of the class $F_{\theta}\left(p,q,\gamma,\beta\right)$ are the functions $f_{p}(z)=z^{p}$ and

$$f_{p+k}(z)=z^{p}-\frac{((p-q)\cos\theta-\beta)\delta(p,q)}{(p-q+k\gamma)\delta(p+k,q)}e^{-i\theta}z^{p+k}\ (k\geq 1). \quad (27)$$

5. Radii of close-to-convexity, starlikeness and convexity

Theorem 7. Let the function $f(z)$ defined by (1) be in the class $.F_\theta\left(p,q,\gamma,\beta\right)$. Then $f(z)$ is p -valent close-to-convex of order $\eta\ (0\le\eta<p-q)\ in\ |z|\le r_1,$ where

$$r_1=\inf_{k\ge1}\left\{\frac{(p-q+k\gamma)\delta(p+k,q)(p-\eta)}{((p-q)\cos\theta-\beta)\delta(p,q)(k+p)}\right\}^{\frac{1}{k}}. \tag{28}$$

The result is sharp, the extremal function given by (6) .

Proof. We must show that

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|\le p-\eta\ for\ |z|\le r_1, \tag{29}$$

where r_1 is given by (5.1) . Indeed we find from (1) that

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|\le\sum_{k=1}^{\infty}(p+k)a_{p+k}|z|^k.$$

Thus

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|\le p-\eta,$$

if

$$\sum_{k=1}^{\infty}\left(\frac{k+p}{p-\eta}\right)a_{p+k}|z|^k\le1. \tag{30}$$

But by using Theorem 1, (30) will be true if

$$\left(\frac{k+p}{p-\eta}\right)|z|^k\le\left(\frac{(p-q+k\gamma)\delta(p+k,q)}{((p-q)\cos\theta-\beta)\delta(p,q)}\right).$$

Then

$$|z|\le\left\{\frac{(p-q+k\gamma)\delta(p+k,q)(p-\eta)}{((p-q)\cos\theta-\beta)\delta(p,q)(k+p)}\right\}^{\frac{1}{k}}. \tag{31}$$

The result follows easily from (31) .

Theorem 8. Let the function $f(z)$ defined by (1) be in the class $F_\theta\left(p,q,\gamma,\beta\right)$. Then $f(z)$ is p -valent starlike of order $\eta\ (0\le\eta<p-q)\ in\ |z|\le r_2,$ where

$$r_2=\inf_{k\geq 1}\left\{\frac{(p-q+k\gamma)\delta(p+k,q)(p-\eta)}{((p-q)\cos\theta-\beta)\delta(p,q)(k+p-\eta)}\right\}^{\frac{1}{k}}. \qquad (32)$$

The result is sharp, the extremal function given by (6) .

Proof. We must show that

$$\left|\frac{zf'(z)}{f(z)}-p\right|\leq p-\eta \ for \ |z|\leq r_2, \qquad (33)$$

where r_2 is given by (32) . Indeed we find from the definition of (1) that

$$\left|\frac{zf'(z)}{f(z)}-p\right|\leq\frac{\sum_{k=1}^{\infty}ka_{p+k}|z|^k}{1-\sum_{k=1}^{\infty}a_{p+k}|z|^k}.$$

Thus

$$\left|\frac{zf'(z)}{f(z)}-p\right|\leq p-\eta,$$

if

$$\sum_{k=1}^{\infty}\left(\frac{k+p-\eta}{p-\eta}\right)a_{p+k}|z|^k\leq 1. \qquad (34)$$

But by using Theorem 1, (34) will be true if

$$\left(\frac{k+p-\eta}{p-\eta}\right)|z|^k\leq\left(\frac{(p-q+k\gamma)\delta(p+k,q)}{((p-q)\cos\theta-\beta)\delta(p,q)}\right).$$

Then

$$|z|\leq\left\{\frac{(p-q+k\gamma)\delta(p+k,q)(p-\eta)}{((p-q)\cos\theta-\beta)\delta(p,q)(k+p-\eta)}\right\}^{\frac{1}{k}}. \qquad (35)$$

The result follows easily from (35) .

Corollary 3. Let the function $f(z)$ defined by (1) be in the class $\cdot$ Then $f(z)$ is in p -valent convex of order η $(0\leq\eta<p-q)$ in $|z|\leq r_3,$ where

$$r_3=\inf_{k\geq 1}\left\{\frac{p(p-q+k\gamma)(p-\eta)\delta(p+k,q)}{(p+k)(k+p-\eta)((p-q)\cos\theta-\beta)\delta(p,q)}\right\}^{\frac{1}{k}}. \qquad (36)$$

The result is sharp, with the extremal function given by (6).

6. Modified Hadamard products

For the functions $f_j(z)\,(j=1,2)$ defined by (17) and belonging to the class $T_p(\theta)$, the modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z^p - \sum_{k=1}^{\infty} a_{p+k,1} a_{p+k,2} z^{p+k}. \qquad (37)$$

Theorem 9. Let the functions $f_j(z)\,(j=1,2)$ defined by (17) be in the class $F_\theta\left(p,q,\gamma,\beta\right)$. Then $(f_1 * f_2)(z) \in F_{2\theta}\left(p,q,\gamma,\alpha\right)$, where

$$\alpha = (p-q)\cos 2\theta - \frac{((p-q)\cos\theta-\beta)^2\delta(p,q)}{(p-q+k\gamma)\delta(p+k,q)}. \qquad (38)$$

The result is sharp for the functions $f_j(z)$ given by

$$f_j(z) = z^p - \left(\frac{((p-q)\cos\theta-\beta)(p-q+1)}{(p-q+\gamma)(p+1)}\right)e^{-i\theta} z^{p+1}\ (j=1,2). \qquad (39)$$

Proof. Employing the technique used ealier by Schild and Silverman [6]. We need only to find the largest ؏ such that

$$\sum_{k=1}^{\infty} e^{2i\theta}\left(\frac{(p-q+k\gamma)\delta(p+k,q)}{((p-q)\cos 2\theta-\alpha)\delta(p,q)}\right)a_{p+k,1}a_{p+k,2} \le 1. \qquad (40)$$

Since $f_j(z)\,(j=1,2)$ are in the class $F_\theta\left(p,q,\gamma,\beta\right)$, it follows from Theorem 1, that

$$\sum_{k=1}^{\infty} e^{i\theta}\left(\frac{(p-q+k\gamma)\delta(p+k,q)}{((p-q)\cos\theta-\beta)\delta(p,q)}\right)a_{p+k,j} \le 1, \qquad (41)$$

for every $j=1,2.$ By the Cauchy Schwarz inequality we have

$$\sum_{k=1}^{\infty} e^{i\theta}\left(\frac{(p-q+k\gamma)\delta(p+k,q)}{((p-q)\cos\theta-\beta)\delta(p,q)}\right)\sqrt{a_{p+k,1}a_{p+k,2}} \le 1. \qquad (42)$$

Therefore, (40) will be satisfied if

$$\sum_{k=1}^{\infty} e^{2i\theta}\left(\frac{(p-q+k\gamma)\delta(p+k,q)}{((p-q)\cos 2\theta-\alpha)\delta(p,q)}\right)a_{p+k,1}a_{p+k,2} \le$$

$$\sum_{k=1}^{\infty} e^{i\theta}\left(\frac{(p-q+k\gamma)\delta(p+k,q)}{((p-q)\cos\theta-\beta)\delta(p,q)}\right)\sqrt{a_{p+k,1}a_{p+k,2}}.$$

Then

$$\sqrt{a_{p+k,1}a_{p+k,2}} \le \left(\frac{(p-q)\cos 2\theta - \alpha}{(p-q)\cos\theta - \beta} \right) e^{-i\theta}. \tag{43}$$

Since (42) implies

$$\sqrt{a_{p+k,1}a_{p+k,2}} \le \left(\frac{((p-q)\cos\theta - \beta)\delta(p,q)}{(p-q+k\gamma)\delta(p+k,q)} \right) e^{-i\theta}, \tag{44}$$

then form (43) and (44) we have

$$\alpha \le (p-q)\cos 2\theta - \frac{((p-q)\cos\theta - \beta)^2 \delta(p,q)}{(p-q+k\gamma)\delta(p+k,q)}. \tag{45}$$

Now defining the function $G(k)$ by

$$G(k) = (p-q)\cos 2\theta - \frac{((p-q)\cos\theta - \beta)^2 \delta(p,q)}{(p-q+k\gamma)\delta(p+k,q)}, \tag{46}$$

we see that $G(k)$ is an increasing function of $k(k \in \mathrm{N})$. Therefore, we conclude that

$$\alpha \le G(1) = (p-q)\cos 2\theta - \frac{((p-q)\cos\theta - \beta)^2 (p-q+1)}{(p-q+\gamma)(p+1)}, \tag{47}$$

which evidently completes the proof of Theorem 9 .

Using arguments similiar to those in the proof of Theorem 9 , we obtain the following theorem.

Theorem 10. Let the function $f_1(z)$ defined by (17) be in the class $F_\theta\left(p,q,\gamma,\beta\right)$ and the function $f_2(z)$ defined by (17) be in the class $F_\theta\left(p,q,\gamma,\varphi\right)$. Then $(f_1 * f_2)(z) \in F_{2\theta}\left(p,q,\gamma,\zeta\right)$, where

$$\zeta = (p-q)\cos 2\theta - \frac{((p-q)\cos\theta - \beta)((p-q)\cos\theta - \varphi))(p-q+1)}{(p-q+\gamma)(p+1)}. \tag{49}$$

The result is sharp for the functions $f_j(z)\,(j=1,2)$ given by

$$f_1(z) = z^p - \left(\frac{((p-q)\cos\theta - \beta)(p-q+1)}{(p-q+\gamma)(p+1)} \right) e^{-i\theta} z^{p+1}, \tag{50}$$

and

$$f_2(z) = z^p - \left(\frac{((p-q)\cos\theta - \varphi)(p-q+1)}{(p-q+\gamma)(p+1)} \right) e^{-i\theta} z^{p+1}. \tag{51}$$

Theorem 11. Let the functions $f_j(z)\,(j=1,2)$ defined by (17) be in the class $F_\theta\left(p,q,\gamma,\beta\right)$. Then the function

$$h(z)=z^p-\sum_{k=1}^{\infty}(a_{p+k,1}^2+a_{p+k,2}^2)z^{p+k}, \tag{52}$$

belongs to the class $F_{2\theta}\left(p,q,\gamma,\eta\right)$, where

$$\eta(p;q,\beta,\gamma;\theta)=(p-q)\cos 2\theta-\frac{2((p-q)\cos\theta-\beta)^2\delta(p,q)}{(p-q+k\gamma)\delta(p+k,q)}. \tag{53}$$

The result is sharp for the functions $f_j(z)$ is given by (39).

Proof . By using Theorem 1, we have

$$\sum_{k=1}^{\infty}\left[e^{i\theta}\left(\frac{(p-q+k\gamma)\delta(p+k,q)}{((p-q)\cos\theta-\beta)\delta(p,q)}\right)\right]^2 a_{p+k,1}^2\le$$

$$\left[\sum_{k=1}^{\infty}e^{i\theta}\left(\frac{(p-q+k\gamma)\delta(p+k,q)}{((p-q)\cos\theta-\beta)\delta(p,q)}\right)a_{p+k,1}\right]^2\le 1, \tag{54}$$

and

$$\sum_{k=1}^{\infty}\left[e^{i\theta}\left(\frac{(p-q+k\gamma)\delta(p+k,q)}{((p-q)\cos\theta-\beta)\delta(p,q)}\right)\right]^2 a_{p+k,2}^2\le$$

$$\left[\sum_{k=1}^{\infty}e^{i\theta}\left(\frac{(p-q+k\gamma)\delta(p+k,q)}{((p-q)\cos\theta-\beta)\delta(p,q)}\right)a_{p+k,2}\right]^2\le 1. \tag{55}$$

It follow from (54) and (55) that

$$\sum_{k=1}^{\infty}\frac{1}{2}\left[e^{i\theta}\left(\frac{(p-q+k\gamma)\delta(p+k,q)}{((p-q)\cos\theta-\beta)\delta(p,q)}\right)\right]^2(a_{p+k,1}^2+a_{p+k,2}^2)\le 1. \tag{56}$$

Therefore, we need to find the largest $\eta=\eta(p;q,\beta,\gamma;\theta)$ such that

$$\sum_{k=1}^{\infty}e^{2i\theta}\frac{(p-q+k\gamma)\delta(p+k,q)}{((p-q)\cos 2\theta-\eta)\delta(p,q)}(a_{p+k,1}^2+a_{p+k,2}^2)\le 1. \tag{57}$$

From (56) and (57) , we have

$$\sum_{k=1}^{\infty} e^{2i\theta} \frac{(p-q+k\gamma)\delta(p+k,q)}{((p-q)\cos 2\theta-\eta)\delta(p,q)}(a_{p+k,1}^2+a_{p+k,2}^2) \le$$

$$\sum_{k=1}^{\infty} \frac{1}{2}\left[e^{i\theta}\left(\frac{(p-q+k\gamma)\delta(p+k,q)}{((p-q)\cos\theta-\beta)\delta(p,q)}\right)\right]^2 (a_{p+k,1}^2+a_{p+k,2}^2), \tag{58}$$

that is

$$\eta \le (p-q)\cos 2\theta - \frac{2((p-q)\cos\theta-\beta)^2\delta(p,q)}{(p-q+k\gamma)\delta(p+k,q)}. \tag{59}$$

Since

$$D(k) = (p-q)\cos 2\theta - \frac{2((p-q)\cos\theta-\beta)^2\delta(p,q)}{(p-q+k\gamma)\delta(p+k,q)}, \tag{60}$$

is an increasing function of $k\ (k \in \mathrm{N})$, we obtain

$$\eta \le D(1) = (p-q)\cos 2\theta - \frac{2((p-q)\cos\theta-\beta)^2(p-q+1)}{(p-q+\gamma)(p+1)}, \tag{60}$$

and hence the proof of Theorem 11 is completed.

Remark.6.1.

(1) Putting $\theta = 0,$ $p = 1$ and $q = 0$ in our results, we obtain the results obtained by Bhoosnurmath and Swamy [3];

(2) Putting $\gamma = p = 1$ and $q = 0$ in our results, we obtain the results obtained by Sekine [8].

7. Definitions and applications of fractional calculus

Many essentially equivalent definitions of fractional calculus (that is, fractional derivatives and fractional integrals) have been given in the literature (cf., e.g. [1], [10] and [11]. We find it to be convenient to recall here the following definitions which were used recently by Owa [5] and by Srivastava and Owa [8]).

Definition 1. The fractional integral of order μ is defined, for a function $f(z),$ by

$$D_z^{-\mu} f(z) = \frac{1}{\Gamma(\mu)}\int_0^z \frac{f(t)}{(z-t)^{1-\mu}}\,dt\ (\mu > 0), \tag{61}$$

where $f(z)$ is an analytic function in a simply-connected region of the complex $z-$ plane containing the origin and the multiplicity of $(z-t)^{\mu-1}$ is removed by requiring $\log(z-t)$ to be real when $z-t>0$.

Definition 2. The fractional derivative of order μ is defined, for a function $f(z)$, by

$$D_z^{\mu} f(z) = \frac{1}{\Gamma(1-\mu)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^{\mu}} dt \; (0 \le \mu < 1), \tag{62}$$

where $f(z)$ is an analytic function in a simply-connected region of the complex $z-$ plane containing the origin and the multiplicity of $(z-t)^{-\mu}$ is removed by requiring $\log(z-t)$ to be real when $z-t>0$.

Definition 3. Under the hypotheses of definition 2, the fractional derivative of order $n+\mu$ is defined by

$$D_z^{n+\mu} f(z) = \frac{d^n}{dz^n} D_z^{\mu} f(z) \; (0 \le \mu < 1; \; n \in \mathrm{N}_0 \; . \tag{63}$$

Theorem 12. Let the function $f(z)$ defined by (1) be in the class $F_\theta\left(p,q,\gamma,\beta\right)$. Then we have

$$\left|D_z^{-\mu} f^{(q)}(z)\right| \ge \frac{\Gamma(p-q+1)}{\Gamma(p-q+\mu+1)} |z|^{p-q+\mu} \times \\ \times \left\{ \delta(p,q) - \frac{(p-q+1)((p-q)\cos\theta-\beta)\delta(p,q)}{(p-q+\gamma)(p+\mu+1)} |z| \right\}, \tag{64}$$

and

$$\left|D_z^{-\mu} f^{(q)}(z)\right| \le \frac{\Gamma(p-q+1)}{\Gamma(p-q+\mu+1)} |z|^{p-q+\mu} \times \\ \times \left\{ \delta(p,q) + \frac{(p-q+1)((p-q)\cos\theta-\beta)\delta(p,q)}{(p-q+\gamma)(p+\mu+1)} |z| \right\}, \tag{65}$$

for $\mu>0$ and $z \in U$. The result is sharp.

Proof. Let

$$F(z) = \frac{\Gamma(p-q+\mu+1)}{\Gamma(p-q+1)} z^{-\mu} D_z^{-\mu} f^{(q)}(z) = \delta(p,q) z^{p-q} - \\ - \sum_{k=1}^{\infty} \frac{\Gamma(p-q+k+1)\Gamma(p-q+\mu+1)}{\Gamma(p-q+1)\Gamma(p-q+k+\mu+1)} \delta(p+k,q) a_{p+k} z^{p-q+k}.$$

Then

$$F(z) = \delta(p,q) z^{p-q} - \sum_{k=1}^{\infty} \Psi(k) \delta(p+k,q) a_{p+k} z^{p+k-q}, \tag{66}$$

where

$$\Psi(k)=\frac{\Gamma(p-q+k+1)\Gamma(p-q+\mu+1)}{\Gamma(p-q+1)\Gamma(p-q+k+\mu+1)}\ (\mu>0).$$

Since $\Psi(k)$ is an decreasing function of $k\ (k\in \mathrm{N})$, then

$$0<\Psi(k)\le\Psi(1)=\frac{(p-q+1)}{(p-q+\mu+1)}. \tag{67}$$

Also, according to Theorem 1, we have

$$\frac{(p-q+\gamma)(p+1)}{((p-q)\cos\theta-\beta)(p-q+1)}\sum_{k=1}^{\infty}a_{p+k}\le\sum_{k=1}^{\infty}\left(\frac{(p-q+k\gamma)\delta(p+k,q)}{((p-q)\cos\theta-\beta)\delta(p,q+1)}\right)a_{p+k}\le 1.$$

Then

$$\sum_{k=1}^{\infty}a_{p+k}\le\frac{((p-q)\cos\theta-\beta)(p-q+1)}{(p-q+\gamma)(p+1)}. \tag{68}$$

From (7.6) and (7.7), we have

$$|F(z)|\ge\delta(p,q)|z|^{p-q}-\Psi(1)|z|^{p-q+1}\delta(p,q+1)\sum_{k=1}^{\infty}a_{p+k}. \tag{69}$$

In view of (66) and (67), we have

$$|F(z)|=\left|\frac{\Gamma(p-q+\mu+1)}{\Gamma(p-q+1)}z^{-\mu}D_z^{-\mu}f^{(q)}(z)\right|$$

$$\ge\delta(p,q)|z|^{p-q}-\frac{(p-q+1)((p-q)\cos\theta-\beta)\delta(p,q)}{(p-q+\gamma)(p-q+\mu+1)}|z|^{p-q+1},$$

and

$$|F(z)|=\left|\frac{\Gamma(p-q+\mu+1)}{\Gamma(p-q+1)}z^{-\mu}D_z^{-\mu}f^{(q)}(z)\right|$$

$$\le\delta(p,q)|z|^{p-q}+\frac{(p-q+1)((p-q)\cos\theta-\beta)\delta(p,q)}{(p-q+\gamma)(p-q+\mu+1)}|z|^{p-q+1}.$$

which proves the inequalities of Theorem 12. Further equalities are attained for the function

$$D_z^{-\mu}f(z)=\frac{\Gamma(p-q+1)}{\Gamma(p-q+\mu+1)}|z|^{p-q+\mu}\times$$
$$\times\left\{\delta(p,q)-\frac{(p-q+1)((p-q)\cos\theta-\beta)\delta(p,q)}{(p-q+\gamma)(p+\mu+1)}|z|\right\}, \tag{70}$$

or

$$f(z)=z^{p}-\frac{((p-q)\cos\theta-\beta)\delta(p,q)}{(p-q+k\gamma)\delta(p+k,q)}e^{-i\theta}z^{p+1}(z=\pm|z|e^{i\theta}). \qquad (71)$$

Using arguments similiar to those in the proof of Theorem 12, we obtain the following theorem.

Theorem 13. Let the function $f(z)$ defined by (1) be in the class $F_{\theta}\left(p,q,\gamma,\beta\right)$. Then we have

$$\left|D_z^{\mu}f(z)\right|\geq\frac{\Gamma(p-q+1)}{\Gamma(p-q-\mu+1)}|z|^{p-q+\mu}\times$$
$$\times\left\{\delta(p,q)-\frac{(p-q+1)((p-q)\cos\theta-\beta)\delta(p,q)}{(p-q+\gamma)(p-\mu+1)}|z|\right\}, \qquad (72)$$

and

$$\left|D_z^{\mu}f(z)\right|\leq\frac{\Gamma(p-q+1)}{\Gamma(p-q-\mu+1)}|z|^{p-q+\mu}\times$$
$$\times\left\{\delta(p,q)+\frac{(p-q+1)((p-q)\cos\theta-\beta)\delta(p,q)}{(p-q+\gamma)(p-\mu+1)}|z|\right\}, \qquad (73)$$

for $0\leq\mu<1$ and $z\in U$. The result is sharp for the function $f(z)$ given by (71).

Remarks.

(1) Putting $q=0$ in our results, we obtain the results obtained by El-Ashwah et al. [4];

(2) Putting $\theta=q=0$ in our results, we obtain the results obtained by Aouf and Darwish [2] and Lee et al. [5].

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Yüksək tərtib törəməli və mənfi əmsalli *p*-valent funksiyalarin ümumiləşmiş sinfi

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XÜLASƏ

Işdə mənfi əmsallı *p*-valent funksiyaların yeni $F_\theta(p,q,\gamma,\beta)$ sinfi nəzərdən keçirilir. Burada əmsallardan düzəldilmiş bərabərsizliklər, distorsiya teoremləri, ekstremal nöqtələr və qabarıqlılığa yaxın radiuslar, bu sinifdən olan funksiyalar üçün qabarıqlılıq şərtləri alınmışdır. Həmçinin $F_\theta(p,q,\gamma,\beta)$ sinfindən olan bir neçə funksiyaların modifikasiya edilmiş Hadamar hasilləri alınmışdır. Sonda kəsr hesabını özündə saxlayan bir neçə distorsiya bərabərsizlikləri də araşdırılmışdır.

Açar sözlər: Analitik p-valent funksiyaları, Hadamar hasili, kəsr hesabı operatorları.

Обобщенный класс *p*-валентных функций с производными высшего порядка и негативными коэффициентами

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РЕЗЮМЕ

В этой работе изучили новый класс p -валентных функций с отрицательными коэффициентами. Мы получили неравенства коэффициентов, теоремы искажения, экстремальные точки и радиус близких к выпуклости, звездность и выпуклость для функций этого класса. Получены также модифицированные произведения Адамара нескольких функций, принадлежащих классу $F_\theta\left(p,q,\gamma,\beta\right)$. Далее, исследовано несколько неравенств искажений, включающих дробное исчисление.

Ключевые слова: Аналитические *p*-валентные функции, произведение Адамара, операторы дробного исчисления.